UNIT OF STUDY-4

GSE GEOMETRY MATHEMATICS TEACHER RESOURCE GUIDE

CIRCLES AND VOLUME:

(January 4- February 24)

This unit deals with all concepts related to circles and volumes of solid figures. By the conclusion of this unit, students should be able to demonstrate the following competencies: Select an appropriate theorem or formula to use to solve a variety of situations involving circles, their segments, and the angles created, as well as volumes of such solids as the cylinder, cone, pyramid, and sphere.

- Construct Inscribed and Circumscribed Circles of triangles
- Complete a Formal Proof of the opposite angles of an Inscribed Quadrilateral being supplementary.
- Find the Arc Length and Area of any sector of a circle.
- Use Cavalier’s Principle to show that the Volume of an Oblique solid can be found using Right Solids.

Table of Contents

	Duration: 16 days (maximum of instructional days on an A/B schedule)
Standards Addressed	**Georgia Standards of Excellence - Mathematics** **Content Standards** (Cluster emphasis is indicated by the following icons: Please note that 70% of the time should be focused on the Major Content. ◊ Major Content □Supporting Content **O**Additional Content) **<u>Understand and apply theorems about circles</u>** **O**MGSE9-12.G.C.1 Understand that all circles are similar. **O**MGSE9-12.G.C.2 Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle. **O**MGSE9-12.G.C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle. **O**MGSE9-12.G.C.4 Construct a tangent line from a point outside a given circle to the circle. **<u>Find arc lengths and areas of sectors of circles</u>** **O**MGSE9-12.G.C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. **<u>Explain volume formulas and use them to solve problems</u>** **OMGSE9-12.G.GMD.1** Give informal arguments for geometric formulas. a. Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments. b. Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle. **OMGSE9-12.G.GMD.2** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures. **OMGSE9-12.G.GMD.3 Use** volume formulas for cylinders, pyramids, cones, and

spheres to solve problems.

Visualize relationships between two-dimensional and three-dimensional objects

OMGSE9-12.G.GMD.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Standards for Mathematical Practice

SMP 1. Make sense of problems and persevere in solving them.
While solving problems involving volumes of complex solids, students should be able to make sense of the problem and be able to persevere to break the solids and find individual volumes.

Students:	Because Teachers:
➢ Read the task carefully. ➢ Draw pictures, diagrams, tables, or use objects to make sense of the task. ➢ Discuss the meaning of the task with classmates. ➢ Make choices about which solution path to take. ➢ Try out potential solution paths and make changes as needed. ➢ Check answers and makes sure solutions are reasonable and make sense. ➢ Explore other ways to solve the task. ➢ Persist in efforts to solve challenging tasks, even after reaching a point of frustration.	✓ Provide rich tasks aligned to the standards. ✓ Allow students time to initiate a plan; uses question prompts as needed to assist students in developing a pathway. ✓ Continually ask students if their plans and solutions make sense. ✓ Question students to see connections to previous solution attempts and/or tasks to make sense of current task. ✓ Consistently ask students to defend and justify their solution by comparing solution paths. ✓ Provide appropriate time for students to engage in the productive struggle of problem-solving. ✓ Differentiate to keep advanced students challenged during work time.

SMP 2. Reason abstractly and quantitatively by requiring students to make sense of quantities and their relationships to one another in problem situations.

Students:	Because Teachers:
➢ Use mathematical symbols to represent situations ➢ Take quantities out of context to work with them (decontextualizing) ➢ Put quantities back in context to see if they make sense (contextualizing) ➢ Consider units when determining if the answer makes sense in terms of the situation	✓ Provide a variety of problems in different contexts that allow students to arrive at a solution in different ways ✓ Use think aloud strategies as they model problem solving ✓ Attentively listen for strategies students are using to solve problems

SMP 3. **Construct viable arguments and critique the reasoning of others** by engaging students on discussion of why they agree or disagree with responses, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Students:	Because Teachers:
➢ Make and tests conjectures. ➢ Explain and justifies their thinking using words, objects, and drawings. ➢ Listen to the ideas of others and decides if they make sense. ➢ Ask useful questions. ➢ Identify flaws in logic when responding to the arguments of others. ➢ Elaborate with a second sentence (spontaneously or prompted by the teacher or another student) to explain their thinking and connect it to their first sentence. ➢ Talks about and asks questions about each other's thinking, in order to clarify or improve their own mathematical understanding. ➢ Revise their work based upon the justification and explanations of others.	✓ Encourage students to use proven mathematical understandings, (definitions, properties, conventions, theorems, etc.) to support their reasoning. ✓ Question students so they can tell the difference between assumptions and logical conjectures. ✓ Ask questions that require students to justify their solution and their solution pathway. ✓ Prompt students to respectfully evaluate peer arguments when solutions are shared. ✓ Ask students to compare and contrast various solution methods. ✓ Create various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.).

SMP 4. Model with mathematics.

Students:	Because Teachers:
➢ Use mathematical models (i.e. formulas, equations, symbols) to solve problems in the world ➢ Use appropriate tools such as objects, drawings, and tables to create mathematical models ➢ Make connections between different mathematical representations (concrete, verbal, algebraic, numerical, graphical, pictorial, etc.) ➢ Check to see if an answer makes sense within the context of a situation and changing the model as needed	✓ Provide opportunities for students to solve problems in real life contexts ✓ Identify problem solving contexts connected to student interests

5. Use appropriate tools strategically by expecting students to consider available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a compass, a calculator, software, etc.

Students:	Because Teachers:
➢ Use technological tools to explore and deepen understanding of concepts ➢ Decide which tool will best help solve the problem. Examples may include: o Calculator o Concrete models o Digital Technology o Pencil/paper o Ruler, compass, protractor ➢ Estimate solutions before using a tool ➢ Compare estimates to solutions to see if the tool was effective	✓ Make a variety of tools readily accessible to students and allowing them to select appropriate tools for themselves ✓ Help students understand the benefits and limitations of a variety of math tools

SMP 7. Look for and make use of structure.

	<table><tr><th>Students:</th><th>Because Teachers:</th></tr><tr><td>➢ Find structure and patterns in numbers ➢ Find structure and patterns in diagrams and graphs ➢ Use patterns to make rules about math ➢ Use these math rules to help them solve problems</td><td>✓ Provide sense making experiences for all students ✓ Allow students to do the work of using structure to find the patterns for themselves rather than doing this work for students</td></tr></table> **Note:** All of the Standards for Mathematical Practice (SMPs) are critical to students fully and appropriately attending to the content. Not all SMPs will occur in every lesson, however SMPs 1, 3, and 6 should be regularly apparent. ***All SMPs should be taught in tandem with the content standards.***
Enduring Understandings	In order to support deep conceptual learning it is important that student leave this unit experience with the following understandings: - Arc length of a circle is proportional to the radius and to the central angle defining the arc. - Area of a sector is proportional to the square of the radius and the central angle. - Properties within one circle and between two or more circles can be used to solve real-world problems. - Informal arguments can be used to support proofs of geometric formulas. - If two figures have the same height and the same cross sectional area at every level, then they have the same volume (Cavalieri's Principle). - Cavalieri's principle can be used for the formulas for volume of a sphere and other solid figures.

Lesson One Progression

Duration 3-4 days

Focus Standard(s)
Understand and apply theorems about circles MGSE9-12.G.C.1 Understand that all circles are similar.

OMGSE9-12.G.C.2 Identify and describe relationships among inscribed angles, radii, chords, tangents, and secants. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

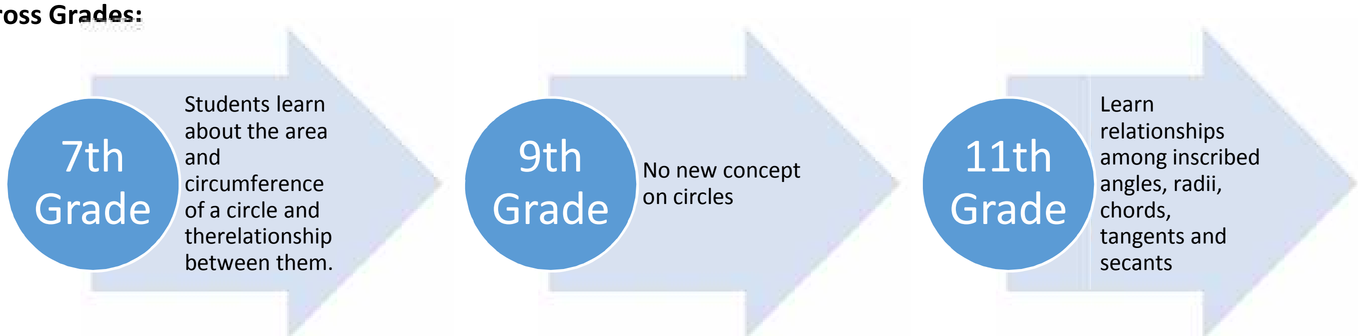
Performance-Based Objectives

As a result of their engagement with this unit…

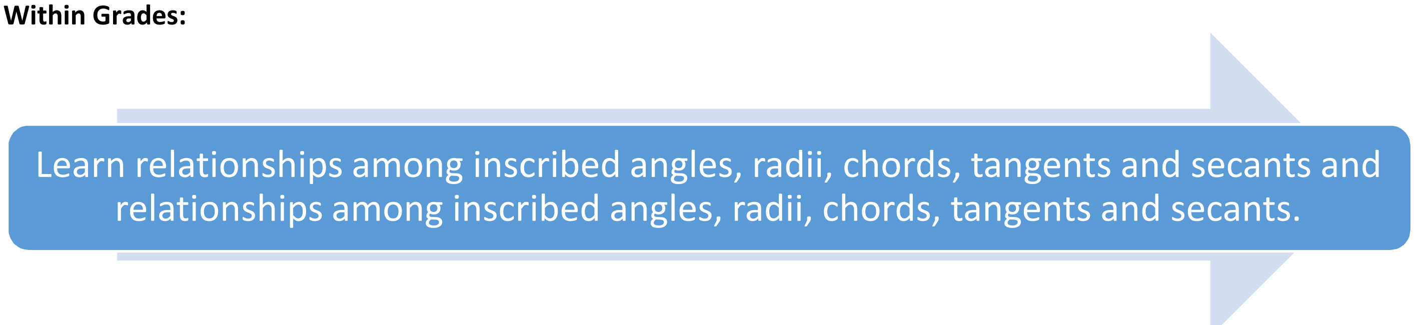
- **MGSE9-12.G.C.1 -SWBAT** understand that circles are similar **IOT** find circumference and areas of circles.
- **MGSE9-12.G.C.2 –SWBAT** apply relationships among inscribed angles, radii, chords, tangents and secants **IOT** solve for unknown quantities in a circle.
- **MGSE9-12.G.C.2 –SWBAT** apply the relationship between central, inscribed, and circumscribed angles **IOT** solve for unknown quantities in a circle.

Building Coherence

Across Grades:



Within Grades:



Key Terms and Definitions

Arc: an unbroken part of a circle; An arc is defined by two endpoints and the points on the circle between those two endpoints.	**Major and Minor Arcs:** Given two points on a circle, the minor arc is the shortest arc linking them. The major arc is the longest. If a circle is divided into two equal arcs, each arc is called a **semicircle**.

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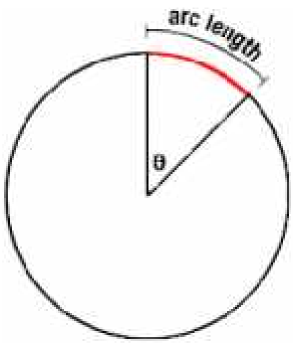
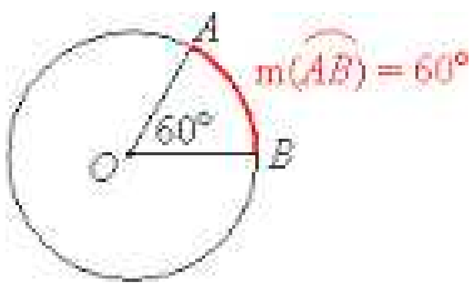
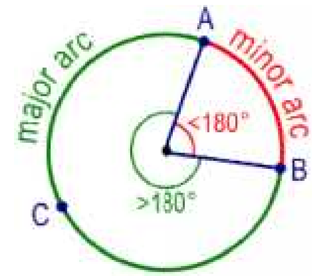
Arcs are measured in two different ways :
Using degree measure and using linear measure. Usually when we refer to the measure of an arc, we are referring to the degree measure. The measure of a minor arc is defined to be the measure of the central angle that intercepts the arc. The measure of a major arc is 360 minus the measure of the minor arc with the same endpoints.
Arc Length: a portion of the circumference of the circle



Arc Measure: The angle that an arc makes at the center of the circle of which it is a part.

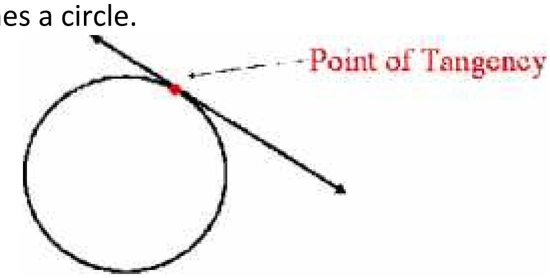


Central Angle: an angle whose vertex is at the center of a circle

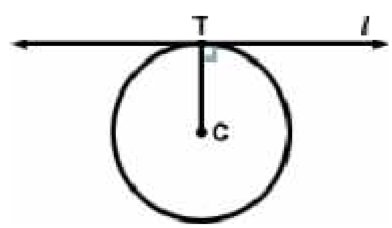


In the above figure, we call the portion of the circle between and including points A and B, arc AB notated by $\overset{\frown}{AB}$. We call the remaining portion of the circle arc ACB, or $\overset{\frown}{ABC}$. Note that major arcs are usually named using three letters in order to distinguish it from the minor arc.

Point of Tangency: the point where a tangent line touches a circle.



Tangent Line: a line in the plane of a circle that intersects a circle at only one point, the point of tangency



Secant Line: a line in the plane of a circle that intersects a circle at exactly two points

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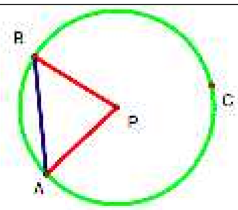
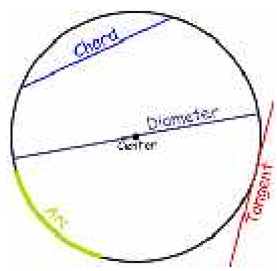
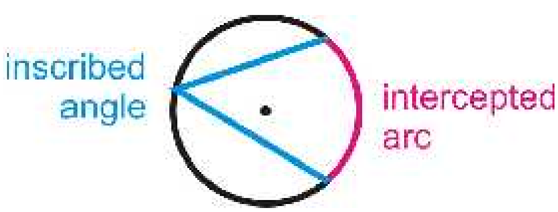


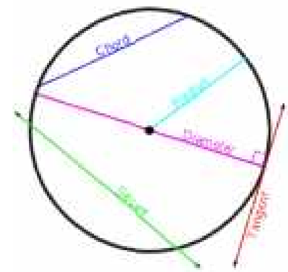
We say that the central angle ∠APB intercepts or has $\overset{\frown}{AB}$.
We also say that $\overset{\frown}{AB}$ subtends or has the central angle ∠APB. Note that when we refer to the arc of a central angle, we usually mean the minor arc unless otherwise stated.

Chord: a segment whose endpoints are on a circle

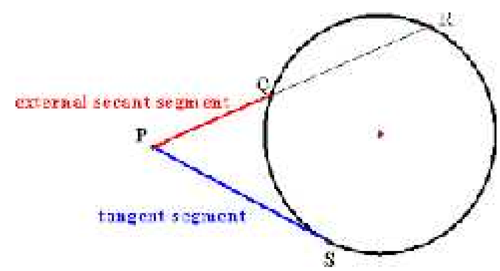


Inscribed Angle: an angle whose vertex is on the circle and whose sides contain chords of a circle





Secant Segment: a segment that contains a chord of a circle and has exactly one endpoint outside of the circle

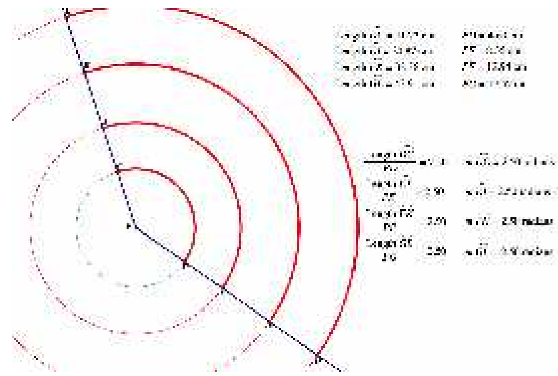


Guiding Question(s)

- How are Congruent Chords related?
- How does the location of the vertex of an angle effect the formula for finding the angle measure?

Interpretations and Reminders

- Teacher can use a geometry software *Geometer's sketchpad or geogebra to draw concentric circles and compare the central angles and arc lengths between two radii. A visual representation like the one on the right should be posted on the wall to remind students that circles are similar.
- Students always get confused between chords and secants. A visual representation of all new vocabulary words posted on wall would help students get familiarized with the academic language.



- All geometric theorems on circles can be visually posted with drawings to help students gain familiarity with them.

Misconceptions

- Students may conclude that circles are not similar because lengths are longer on larger circles.
- Students sometimes confuse inscribed angles and central angles. For example they will assume that the inscribed angle is equal to the arc like a central angle.
- Students may confuse the segment theorems. For example, they will assume that the inscribed angle is equal to the arc like a central angle.
- Students may think they can tell by inspection whether a line intersects a circle in exactly one point. It may be beneficial to formally define a tangent line as the line perpendicular to a radius at the point where the radius intersects the circle.

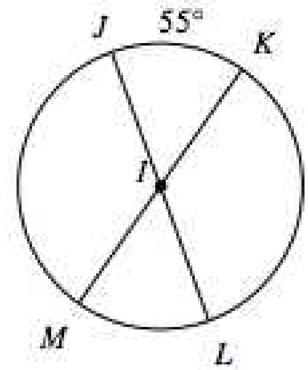
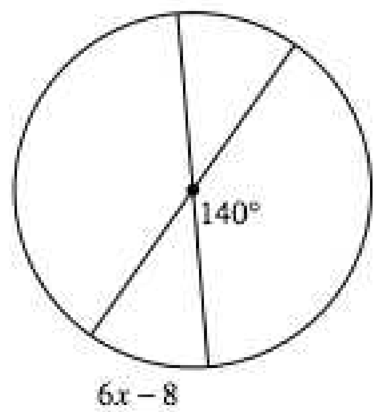
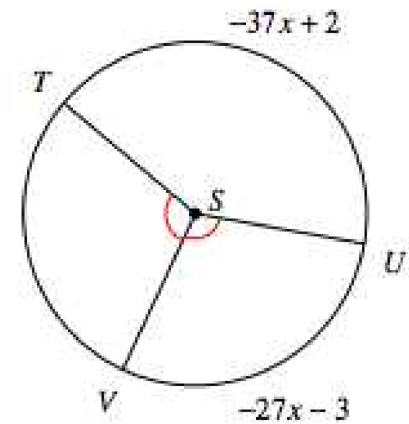
Learning Progression (Suggested Learning Experiences)

Procedural Fluency: (Recommended for 5 - 10 minutes each day: Fluency strategies are useful to activate student voice, solicit prior knowledge and develop fluency based on conceptual understandings.) For additional fluency practice strategies see the table at the end of this document.

Draw and identify the following parts in a circle: arc, minor arc, major arc, and chord.

Graduated Measure (The graduated measure is a quick opportunity to diagnose students' level of comfort with the material before you begin the progression.)

Student Directions: Select a question/problem from the table below that you feel best equipped to answer successfully.

Beginning	Developing	Proficient
Find m <MIJ 	Solve for x: 	Find the indicated measure: $m\angle VST$ 

<u>Focus Lesson</u>: Theorem 1: Circles are similar.

<u>Investigation 1</u>

Start by asking students the following:

How do you define a circle? How do you define the radius and the diameter?

How many measurements does a circle have? What do we need to prove in order to prove that two circles are similar?

Ask students to draw circles of different radii. Draw a diameter as a segment passing through the center of the circle and also draw a radius and measure the radius and the diameter. Also measure the circumference using a piece of string. Complete the table:

Circles	Radius	Diameter	Circumference
Circle 1			
Circle 2			
Circle 3			
Circle 4			

Using the above table, calculate the following ratios:

1. $\frac{D}{R}$______ 2. $\frac{C}{R}$__________ 3. $\frac{C_1}{D}$__________

Spark a discussion amongst students about their observations. Explain to them that since the diameter and circumference are dependent on the radius, dilating and changing the size of the radius can change the size of the circle. Thus only the size is hanged and not the shape. Since the new circle is the result of dilation, the two circles are similar.

Also remind students to use proportionality for defining similarity. Since circles do not have sides, the ratio between the circumference, diameters and radii can be compared to check for similarity. Using the values from the above table, ask students to calculate the following:

1. $\frac{R \qquad 1}{R \qquad 2}$ 2. $\frac{D \qquad\qquad 1}{D \qquad\qquad 2}$ 3. $\frac{C \qquad\qquad\qquad 1}{C \qquad\qquad\qquad 2}$

What do you observe? Is your observation true for any two circles? What do you conclude?
Students should have learned the area formula for circles already. What do you know about $\frac{A \quad o \; C \quad 1}{A \quad o \; C \quad 2}$?

Focus lesson: Theorem 2: Corresponding chords have equal central angles and corresponding arcs have equal central angles.

Before students learn about the relationship between the parts of a circle involving angles and line segments, students should be familiar with a number of new vocabulary words in this lesson. New vocabulary words include major and minor arcs, arc measure (in degrees), semicircle, chord, central angles, inscribed angles, circumscribed angles, tangent, secant, etc.

Also, the relationships or theorems can either be given as an investigative activity using paper, pencil, ruler and protractor or using a software (geometer's sketchpad, geogebra etc.) .

Investigation 2: Using a compass and piece of string, and as an individual:
a. Construct a circle with a radius equal in length to your string. Be sure to mark the center of your circle.
b. Construct a chord on your circle that is approximately 0.25 string lengths.
c. Construct another chord on your circle that is approximately 1.5 string lengths.
d. Connect the center of the circle to the endpoints of the chords, forming two central angles. Measure the central angles.
Record the string length and the measurement of central angles from 3 different students.

Circles	Central angle when the chord is 0.25 string length	Central angle when the chord is 1.5 string length
Circle 1 Radius = String length = ______		

Circle 2 Radius = String length = ______		
Circle 3 Radius = String length = ______		

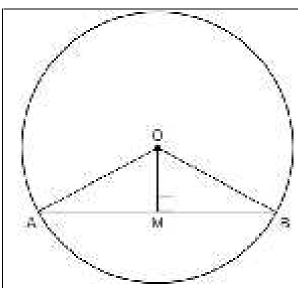
What do you observe about the central angles? Write a conjecture based on your observation.

Teacher Note: *During this activity, students should note that constructing a chord length of a particular string size yields an angle of the same openness regardless of the circle size. Emphasize that when comparing measures, it is necessary to identify that the similarity basis is relative to angle measures. So, corresponding chords have the same central angle, as do corresponding arcs. The converse is also true: corresponding arcs have the same central angle.*

Theorem 3: The perpendicular from the center of a circle to a chord will always bisect the chord (split it into two equal lengths).

Investigation 3

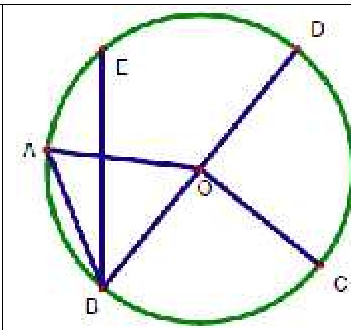
Given OM ⊥AB, prove that OM bisects AB.



Statement	Reason
1. OA = OB 2. <OMA ≅ <OMB= 90^0 3. OM ≅ OM 4. Δ AMO ≅ Δ BMO 5. *A* ≅ BM	1. Radii 2. Definition of perpendicular 3. Reflexive Property 4. SAS Congruence 5. CPCTC
OM bisects AB.	

Refer to the figure at the right.
Identify and name each of the following.
Be sure to use the correct notation.
Two different central angles ∠***DOC, ∠COB, etc.***

b. A minor arc ***arc***
c. A major arc ***arc***
d. A semicircle ***arc***



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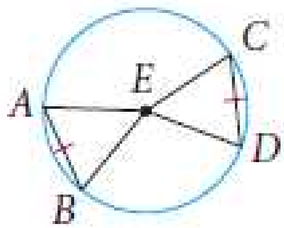
	e. Two different chords f. The central angle subtended by $\overset{\frown}{ABC}$	

Theorem 4: In the same circle or congruent circles, if two chords are congruent, their intercepted arcs are congruent.



Given: Chord AB and Chord CD are Congruent.
Prove: arc AB is congruent to arc CD

Statements	*Reasons*
$\overline{AB} \cong \overline{CD}$	*Given*
$\overline{EA} \simeq \overline{EB} \simeq \overline{EC} \simeq \overline{ED}$	*All radii are congruent*
$\Delta AEB \cong \Delta CED$	*SSS*
$\angle AEB \cong \angle CED$	*Corresponding Parts of Congruent Triangles are Congruent*
$\overset{\frown}{AB} \cong \overset{\frown}{CD}$	*congruent Central Angles create congruent arc measures*

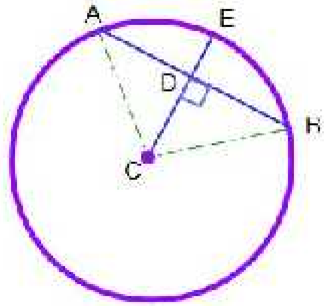
Converse of the above statement:
In the same circle or congruent circles, if two arcs are congruent, then their chords are congruent. Prove.
The above two statements can be combined as a biconditional statement as

Two chords of the same circle or congruent circles are congruent if and only if their intercepted arcs are congruent.

Theorem 5: The radius drawn perpendicular to any chord of a circle bisects the arc intercepted by the chord.

Theorem 5: The radius drawn perpendicular to any chord of a circle bisects the arc intercepted by the chord.

Given CE ⊥ AB, prove that arc AE and arc EB are congruent.



Statement	Reason
1. CA = CB 2. <CDA ≅ <CDB= 90^0 3. OD ≅ OD 4. Δ ADC ≅ Δ BDC 5. <CDA ≅ <CDB 6. Arc AE ≅ Arc EB	1. Radii 2. Definition of perpendicular 3. Reflexive Property 4. SAS Congruence 5. CPCTC 6. Congruent central angles intercept congruent arcs
CE bisects arc AB.	

Theorem 6: Any line that is a perpendicular bisector of a chord of a circle must also contain the center of the circle.

Theorem 6: Any line that is a perpendicular bisector of a chord of a circle must also contain the center of the circle. Prove.

A flow chart proof for the above statement is given below:

If AB is a line segment and P is any point on the perpendicular bisector of AB, then P is equidistant from A and B. ➔PA = PB; however if C is center of the circle, CA = CB and so C lies on the perpendicular bisector of the AB. Thus, the perpendicular bisector of a chord of a circle must also contain the center of the circle.

<u>Guided Practice: Finding the center of a circle given an arc of the circle.</u>

11. An investigator working for the Georgia Bureau of Investigation's crime lab has uncovered a jagged piece of a circular glass plate believed to have been used as a murder weapon. She needs to know the diameter of the plate. How might you use the information you learned in problem 10 to help determine the diameter of the circular plate?
    Use a compass, a straightedge, and a ruler to illustrate your answer.

__COMMENTS__

Students who have not been previously exposed to points of concurrency in triangles will need more support and direct teaching for this part of the unit.

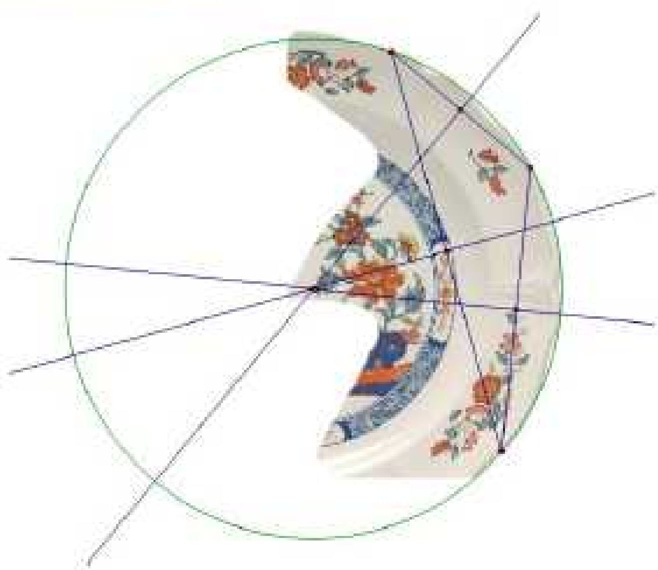
An illustration using an interactive software such as Geometers Sketchpad or Geogebra may be useful. One is included below.

An entire lab day could be created here by providing students with pieces of broken plates and asking them to find the diameter.

__SOLUTION__

By identifying three points on the jagged piece, the investigator could construct a triangle and its three perpendicular bisectors that would meet that the Circumcenter of the triangle. Knowing that the circumcenter is equidistant to the vertices of the triangle and that the circle constructed with these vertices and center is the circumscribed circle will lead the investigator to have the circular plate reconstructed.

The distance from the circumcenter to any one of the triangles vertices will be the radius, which can be doubled to find the diameter of the plate.

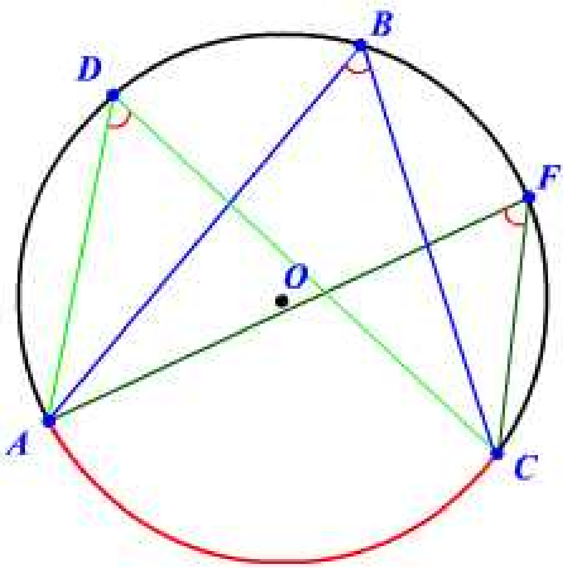


Another version of finding the center of a circle given an arc or three points of the circle is given here:

Constructing a circle through 3 given point

Collaborative Practice:

1. Investigating the relationship between inscribed angles intercepted by the same arc



Measure angles ADC, ABC, and AFC in the above figure using a protractor. What do you observe?

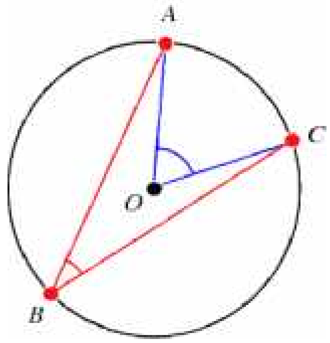
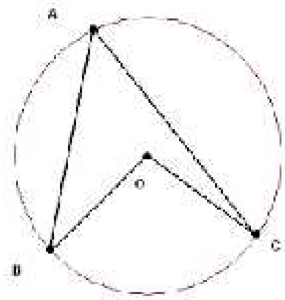
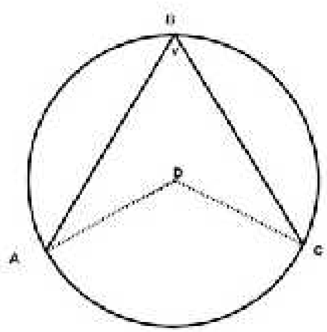
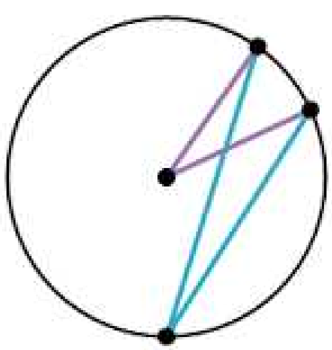
All inscribed angles intercepted by arc AC are congruent.

> All inscribed angles intercepted by the same arc are congruent.

2. Investigating the relationship between inscribed angle and central angle

In each of the circle given below, use a protractor to measure the inscribed angle and the central angle.

	Circle Diagram	Inscribed angle	Central angle	Relationship if any

		m<AOC =	M<ABC =	
				
				
				

Write your conjecture about the relationship between the inscribed angle and the central angle.

Some of the language used in this document is adapted from the CCSS Progressions, NY, Utah, Arizona, Ohio, & Georgia state resources s.

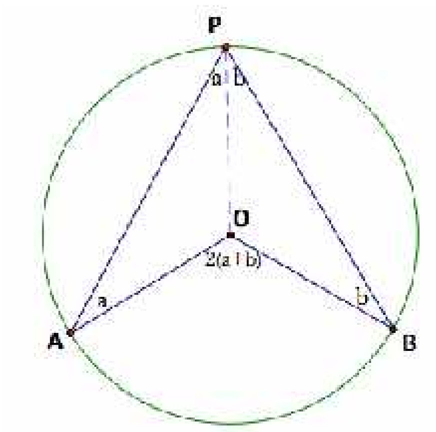
Theorem 7:An inscribed angle is half of the central angle that subtends the same arc.

Inscribed angle = ½ their intercepted arc

https://www.khanacademy.org/math/geometry/hs-geo-circles/hs-geo-inscribed-angles/a/inscribed-and-central-angles-proof

Remind students that a conjecture is not a theorem until proved.

How do you prove the above statement?
Proof:



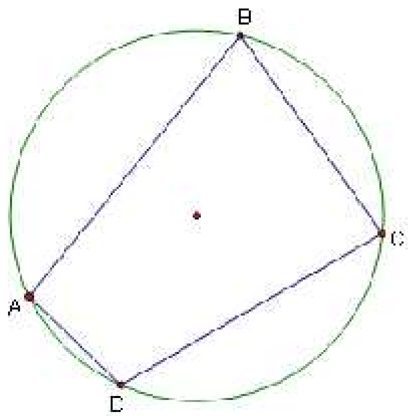
A paragraph proof for the above theorem is provided here:

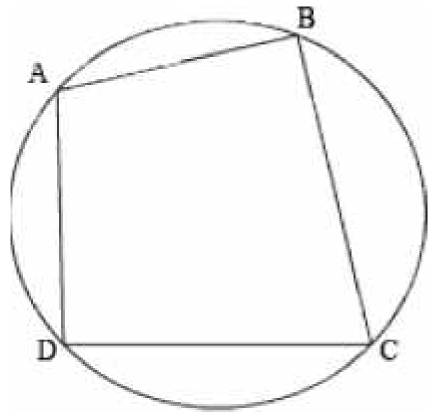
In triangle AOP, ∠AOP = 180-2a, (why?)
In triangle BOP, ∠BOP = 180- 2b (why?)
At point O, (180-2a) +(180-2b) + ∠AOB = 360
∠AOB = 2a+2b = 2 (a+b) = 2 ∠APB

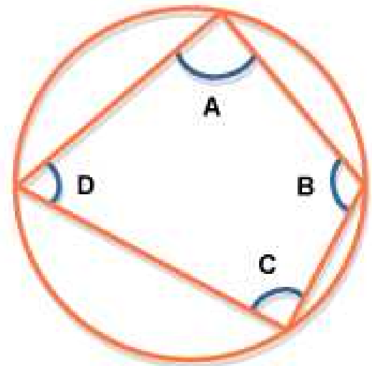
3. Investigating the relationship between angles of a quadrilateral inscribed in a circle.
Before students begin this investigation, remind students the following:

a) *A quadrilateral is a polygon with four sides.*
b) A polygon is **inscribed** in a circle when every vertex of the polygon is on the circle.
c) A quadrilateral inscribed in a circle is also called a cyclic quadrilateral.

Given three quadrilaterals inscribed in a circles, measure the four angles A, B, C and D







Quadrilateral	Angle A	Angle B	Angle C	Angle D
1				
2				
3				

What do you observe? Write your conjecture here.

____
____
____
____
____
____.

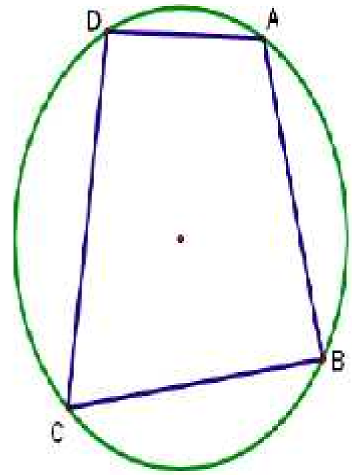
Note: The relationship may not be obvious to some students; teacher has to ask probing questions to steer their thinking towards forming the conjecture.

Theorem 8:

The opposite angles of a quadrilateral inscribed in a circle are supplementary.

Proof to the Theorem:

SOLUTION



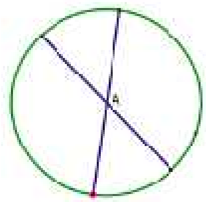
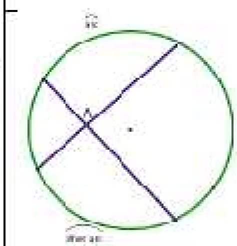
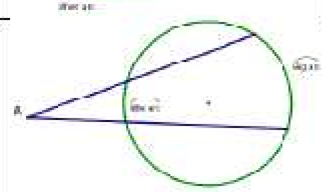
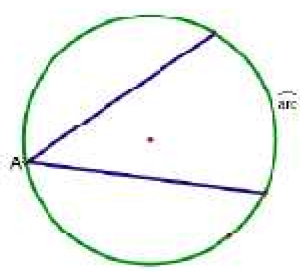
Statements	*Reasons*
$\angle BCD, \angle BAD$ *are inscribed angles*	*given*
$m\angle BCD = \frac{1}{2} m\widehat{BAD}$ $m\angle BAD = \frac{1}{2} m\widehat{BCD}$	*Inscribed Angle Thm*
$m\angle BCD + m\angle BAD = \frac{1}{2}(m\widehat{BAD} + m\widehat{BCD})$	*Addition Property of Equality*
$m\angle BCD + m\angle BAD = \frac{1}{2}(360^\circ)$	*Arc Addition Postulate*
$m\angle BCD + m\angle BAD = 180^\circ$	*multiplication*
$\angle BCD, \angle BAD$ *are supplementary*	*Definition of Supplementary Angles*

4. Investigating the Secant lengths and tangent lengths-Sunrise on New Year Task

Chords Secants and Tangents.docx

Angle between two chords in relation to intercepted arcs

Part 3: Graphic Organizer for Angle Theorems

Location of the Vertex	Picture	Theorem
Inside the circle **At the Center**		$m\angle A = arc$
Not at the center		$m\angle A = \frac{1}{2}(arc + other\ arc)$
Outside of the circle		$m\angle A = \frac{1}{2}(Big\ arc - Little\ arc)$
On the circle		$m\angle A = \frac{1}{2}(arc)$

All of the above theorems and investigations can be demonstrated visually through the use of Geometer's sketchpad. A readily available activity is provided here for the teacher to use for demo purposes with students.

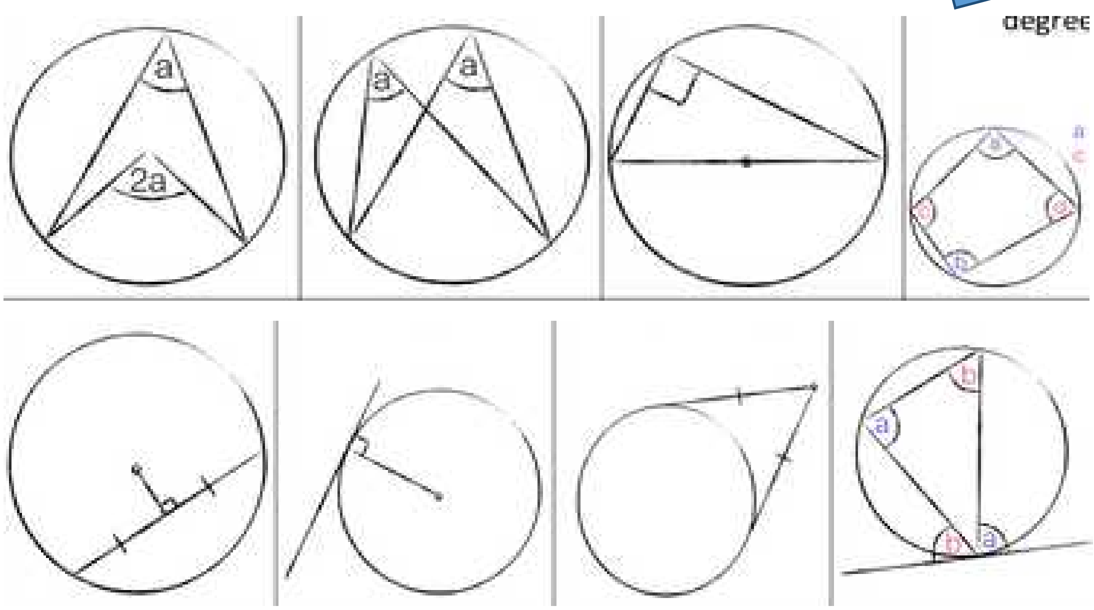

MBCircleAngles.gsp


MBcirclesegments.gsp

Independent Practice

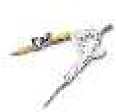
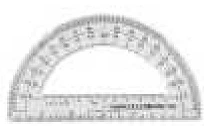
Kuta-Arcs and Central Angles.pdf | Glenco-CirclesG.C 2 (Ang,Arcs)Skills pract | Glencoe Circles-G.C.2(Ins Ang | IM Circles-C 2-Inscribed ang in se

Write the statement of the theorem represented in these visuals:



Unit Assessments –				
Assessment Name	**Assessment**	**Assessment Type**	**Standards Addressed**	**Cognitive Rigor**
Circle Similarity	IM C1- Circle similarity.pdf	Learning Task	MGSE9-12.G.C.1	DOK 2
Neglecting the curvature of the earth	IM C2 Neglecting the curve of earth.pdf	Discovery Task	MGSE9-12.G.C.2, MGSE9-12.G.SRT 8	DOK 3

Differentiated Supports

Learning Difficulty	Students with learning difficulty will benefit from a visual representation of every concept learned related to circles. It would be beneficial to give students graphic organizers for every concept learned and also have visual posters representing the theorems in class. Circle Theorem Poster.docx Glencoe-Circles G.C.2 Intervention.pdf CircleDefinitionsandTheorems.pdf
High-Achieving	Challenge gifted and high achieving students with the following extension tasks: Glencoe-Circles G.C.2 Enrichment.pdf CCSS-CTE-Grain-Storage-Capacity-FINAL.|
English Language Learners	- Have students write their understandings in a math journal. You may want to consider allowing EL students to write notes in their first language and annotate by identifying math specific words they do not have a translation for. - Allow students to create pictorial flash cards for new math words. - Students should use the graphical representations of all circles theorems to express clarity in thought. - Use the sequencing activity to help students determine the order in which to prove theorems using a two-column proof.
Online/Print Resources	
Digital Resources	- https://learnzillion.com/lesson_plans/6602 -Circle Similarity Video - http://www.mathopenref.com/circlecentral.html - https://www.mathsisfun.com/geometry/circle-theorems.html - http://jwilson.coe.uga.edu/emt725/ReviewCir/ReviewCir.htm - https://revisionmaths.com/gcse-maths-revision/shape-and-space/circle-theorems - http://www.mathwarehouse.com/geometry/circle/index.php - http://www.mathwarehouse.com/classroom/free-math-printable-worksheets.php
Print Resources	EngageNY Inscribed and Central ang.pdf angles-and-arcs-formed-by-intersecting-(
Manipulatives /Tools	 Compass  Protractor  Ruler  Sketchpad
Textbook Alignment	
Geometry, McGraw-Hill	pp. 697 – 730

Some of the language used in this document is adapted from the CCSS Progressions, NY, Utah, Arizona, Ohio, & Georgia state resources s.

Lesson Two Progression

Duration 2-3 days

Focus Standard(s)

MGSE9-12.G.C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

MGSE9-12.G.C.4 Construct a tangent line from a point outside a given circle to the circle.

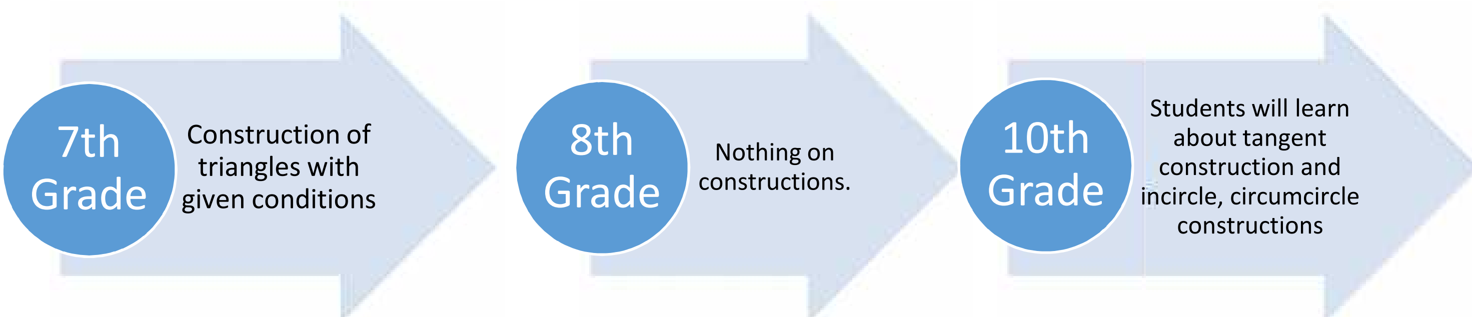
Performance-Based Objectives

As a result of their engagement with this unit…

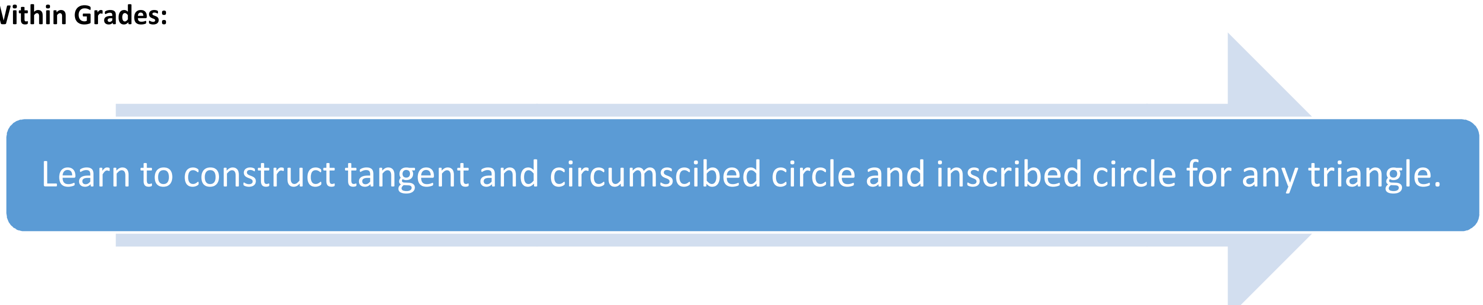
- **MGSE9-12.G.C.3-4 – SWBAT** define the centers of a triangle by constructing inscribed and circumscribed circles **IOT** apply the properties to solve problems.

Building Coherence

Across Grades:

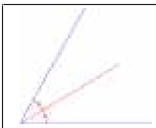


Within Grades:



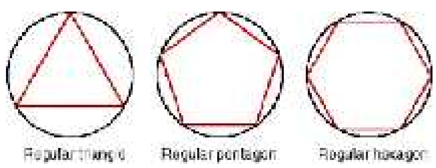
Key Terms and Definitions

Angle Bisector: The angle bisector of an angle is a ray or a line segment that divides the angle into two equal parts.	**Inscribed Circle:** a circle enclosed in a polygon, where every side of the polygon is a tangent to the circle; specifically for this unit the polygon will be a triangle and so the center of the Inscribed Circle is the incenter of the triangle



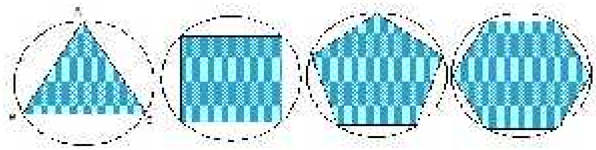
Circumcenter: The point of intersection of the perpendicular bisectors of the sides of a given triangle; the center of the circle circumscribed about a given triangle.

Circumscribed Circle: a circle containing an inscribed polygon; for this unit the polygon will be a triangle and so the center of the circle will be the circumcenter of the triangle.

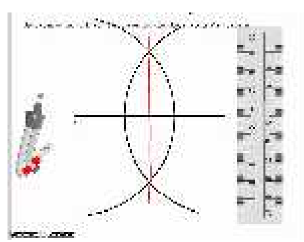


Inscribed: an inscribed planar shape or solid is one that is enclosed by and "fits snugly" inside another geometric shape or solid.

Inscribed Polygon: a polygon whose vertices all lie on a circle



Perpendicular Bisector: A perpendicular bisector of a line segment is a line segment perpendicular to and passing through the midpoint of (left figure). The perpendicular bisector of a line segment can be constructed using a compass by drawing circles centered at and with radius and connecting their two intersections.



Guiding Questions

- How does the GBI use the fact that the perpendicular bisectors of the sides of a triangle are concurrent to solve crimes?
- How did ancient mathematicians construct perfect tangents to a circle, without the software or measuring tools that we have today?

Interpretations and Reminders

- Remind students to use only your compass and straight edge when drawing a construction. No free-hand drawing!
- The Greek mathematician Euclid (also called 'Father of Geometry') wrote about everything learned in high school geometry in his book called 'Elements' 2000 years ago! Unbelievable! This book is still considered as the reference book for geometry. He uses extensive construction techniques to generate hypothesis and then prove them. His construction techniques also helped mathematicians to build formulas and new concepts!

- For those students that ask ‘why learn constructions?’ -Constructing geometrical objects helps in visualizing the concepts and plays a key role in understanding the concept. Students will also be able to make connections between earlier learned concept and the new concept.
- Help students understand the process of construction each step of the way by asking probing questions every step of the way!

Misconceptions

- Students view constructions as a mechanical way of following step-by-step procedures. Remind students that following procedures without internalizing what is being done is a futile exercise!

- Students often do not pay much attention to precision while doing constructions. An angle bisector will not bisect the angle; a perpendicular bisector may not make a right angle if not drawn to precision.

Learning Progression (Suggested Learning Experiences)

Procedural Fluency: (Recommended for 5 - 10 minutes each day: Fluency strategies are useful to activate student voice, solicit prior knowledge and develop fluency based on conceptual understandings.) For additional fluency practice strategies see the table at the end of this document.

Create examples using numerical values for the following theorems:
1. The inscribed angle in a circle is half of the central angle with the same endpoints.
2. If two chords intersect in a circle, the product of the lengths of the segments of one chord equals the product of the segments of the other.

Graduated Measure (The graduated measure is a quick opportunity to diagnose students’ level of comfort with the material before you begin the progression.)

Student Directions: Select a question/problem from the table below that you feel best equipped to answer successfully.

Level 1	Level 2	Level 3
Construct a perpendicular bisector of a line segment.	Draw three perpendicular bisectors of the three sides of a triangle and verify if they are concurrent.	Prove that the three perpendicular bisectors of a triangle are concurrent and give one property of the point of concurrence.

Gradual Release of Responsibility:

In this lesson progression, students will have to learn to construct the following:

1. A tangent to a circle from a point on the circle and from an external point outside of the circle
2. Construct the 3 angle bisectors to construct the incenter and inscribed circle for any kind of triangle
3. Construct the 3 perpendicular bisectors to construct the circumcenter and circumscribed circle for any kind of triangle
4. Learn the properties of the incenter and the circumcenter.

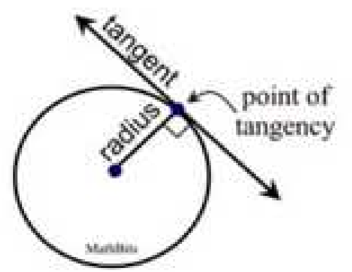
   Note to the Teacher: The above can be done using compass and straight edge (no protractor for angle bisector) and/ or through the use of geometer's sketchpad. Teacher can initially demonstrate the construction using sketchpad tool or using one of the previously completed construction through a video/java app.

Focus Lesson 1: Constructing a tangent line to a given circle at a point:

Tangent to Circle at Point ON Circle

This construction is an easy one if you remember that in a circle, a radius drawn to the point of tangency is perpendicular to the tangent.

Use the construction: construct a perpendicular to a line from a point on the line. This construction is simply a variation of a construction you already know how to draw.



Given: Circle O

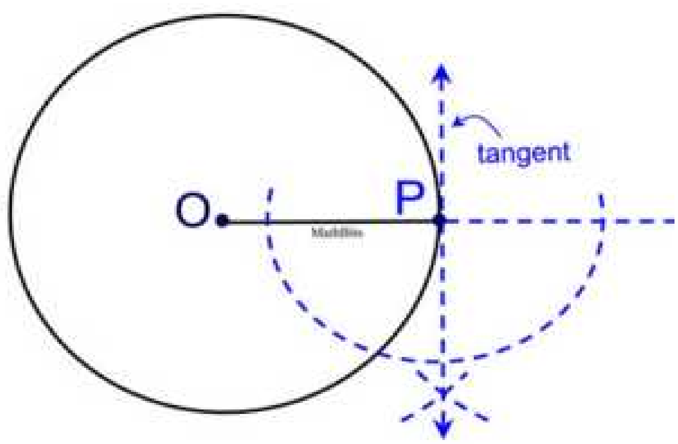
Construct: a tangent to circle at a point on the circle

STEPS:

1. If a point on the circle is not given, draw any radius and label P. If a point is already given on the circle, connect the point to the center of the circle to form a radius.

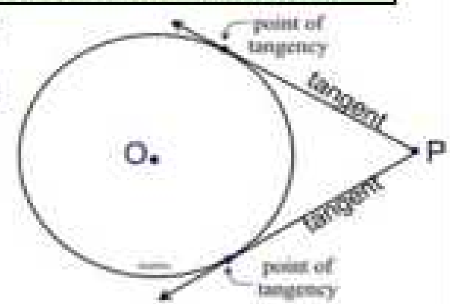
2. Extend the radius past the circle.

3. Construct a perpendicular to the radius line at point P.



Tangent to Circle from Point OUTSIDE Circle (G.C.4)

This construction requires a bit more work. Picture in your mind what these tangents to a circle will look like. The diagram at the right shows two tangents from point P. You need not draw the two tangents, but the two of them may remind you of how this construction will look, as the construction creates two possible tangents.



Given: Circle O

Construct: a tangent to circle O from P

STEPS:

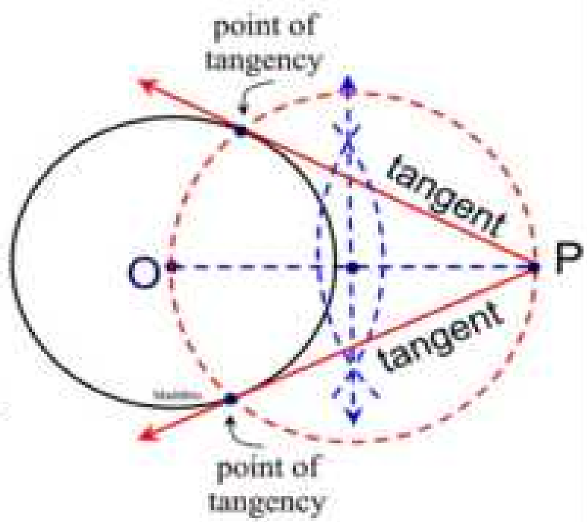
1. Connect O to P.

2. Construct bisector of $\overline{OP}$.

3. Place compass point at midpoint of $\overline{OP}$ and stretch span to O or P.

4. Draw circle.

5. Connect P to where the two circles intersect to create tangents.

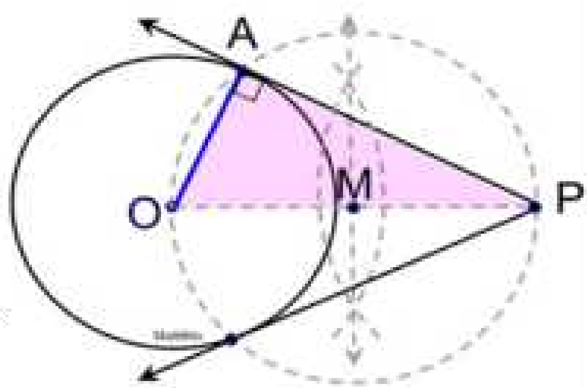


Why does this work?

When the construction is finished, connect O to A to form a radius of circle O.

This radius also forms ΔOAP in circle M. Since $\overline{OP}$ is the diameter of circle M, $\angle OAP$ is an angle inscribed in a semicircle, making it a right angle.

Since $\angle OAP$ is a right angle, $\overline{OA} \perp \overline{AP}$. In circle O, we now have the radius perpendicular to a line passing through a point on the circle (A), making $\overline{AP}$ a tangent to circle O.



The above taken from http://mathbitsnotebook.com/Geometry/Constructions/CCconstructionCircles.html

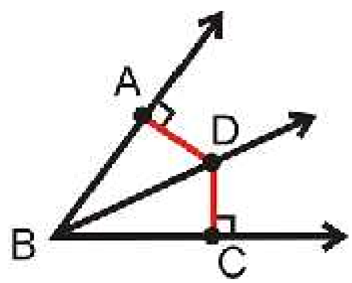
Focus Lesson 2 : Constructing a inscribed circle for a given triangle:

Incircle Construction.pdf

Incenter_GSP.pdf

Circumscribed circle construction.pdf

Collaborative Practice: Constructing a circumscribed circle for a given triangle:

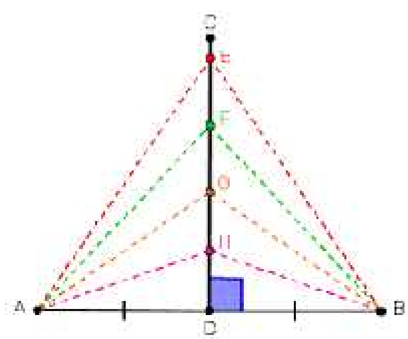


What do you know about the angle bisectors?

The angle bisector is equidistant from the arms of the angle

What do you know about the perpendicular bisector?

The perpendicular bisector is equidistant from the two end points of the line segment



What did you observe about the incenter?

The incenter is equidistant from the three sides of the triangle.

What did you observe about the circumcenter?

The circumcenter is equidistant from the three vertices of the triangle.

Independent Practice:

IM C 2- Rt tri inscribed in circle.pdf

Glencoe-Circles G.C. 2 Word problems.pdf

Additional Unit Assessments –

Assessment Name	Assessment	Assessment Type	Standards Addressed	Cognitive Rigor
Circles in triangles	MAP circles in triangles un.pdf MAP circles in triangles un.pdf MAP-circles_in_triangles_rubric.pdf	Short Cycle task	MGSE9-12.G.C. 3, 4	DOK 3
Circles and Triangles	MAP circles and triangles r1.pdf	Formative Assessment Task	MGSE9-12.G.C.4	DOK 3
Locating a warehouse	IM Locating a warehouse C3.pdf	Formative Assessment Task	MGSE9-12.G.C.3	DOK 3
Placing a Fire Hydrant	IM Placing a fire hydrant C 3.pdf	Formative Assessment Task	MGSE9-12.G.C.3	DOK 3

Differentiated Supports

Learning Difficulty	Students might do well with technological tools rather than paper pencil compass while during constructions. Students should be provided opportunities to explore the theorems using software like sketchpad or geogebra.
High-Achieving Students	High achieving students can be asked to construct the circumcircle for an acute triangle, an obtuse triangle, a right triangle and can be asked to investigate the location of the circumcenter in the three cases. High achieving students can also construct the incenter and circumcenter for an isosceles triangle, an equilateral triangle and

	can write their observations as a conjecture and prove the same.
English Language Learners	- Have students write their understandings in a math journal. You may want to consider allowing EL students to write notes in their first language and annotate by identifying math specific words they do not have a translation for. - Allow students to create pictorial flash cards for new math words. - Students should use the graphical representations of the various centers of triangles to express clarity in thought.
Online/Print Resources	
Digital Resources	- http://www.1728.org/radians.htm - http://www.mathopenref.com/constincircle.html - http://www.mathopenref.com/constcircumcenter.html - http://www.mathopenref.com/consttangent.html - http://mathbitsnotebook.com/Geometry/Constructions/CCconstructionCircles.html - http://mathbits.com/MathBits/GSP/TangentCircle.htm - https://www.youtube.com/watch?v=jwhEgl_lkPM
Print Resources	Tangent Construction .pdf
Manipulatives/Tools	 Compass  Protractor  Ruler  Sketchpad
Textbook Alignment	
Geometry, McGraw-Hill	pp. 723 – 741

Lesson Three Progression

Duration 2-3 days

Focus Standard(s)

Find arc lengths and areas of sectors of circles

OMGSE9-12.G.C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

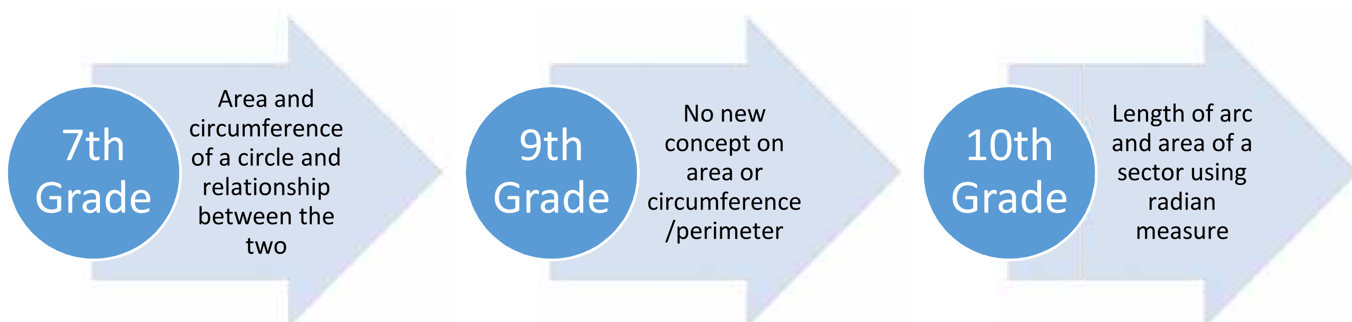
Performance-Based Objectives

As a result of their engagement with this unit…

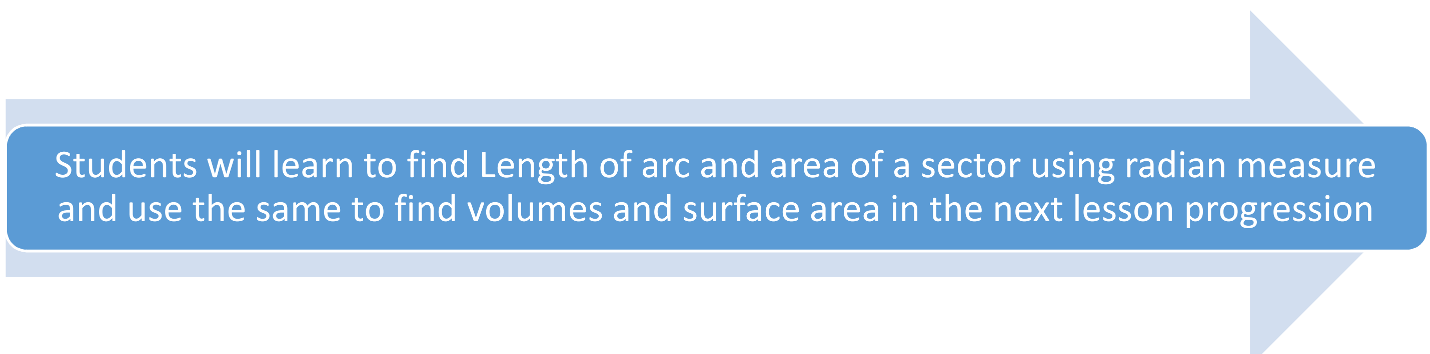
MGSE9-12.G.C.5 – SWBAT define radian measure and derive the formula for area of a sector **IOT** solve circle problems.

Building Coherence

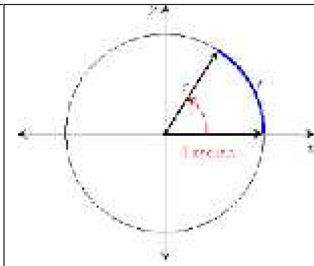
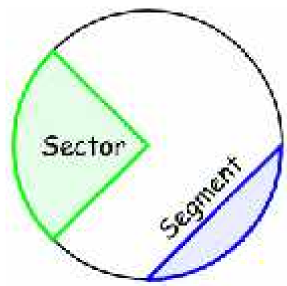
Across Grades:



Within Grades:



Key Terms and Definitions

Arc: an unbroken part of a circle; An arc is defined by two endpoints and the points on the circle between those two endpoints. **Arc Length:** a portion of the circumference of the circle **Radian:** A unit of measure for angles. One radian is the angle made at the center of a circle by an arc whose length is equal to the radius of the circle	 **Sector:** the region bounded by two radii of the circle and their intercepted arc 

Guiding Questions

- How is a radian defined?
- How are the length of the arc of a sector and the area of a sector related?
- Where is Arc Length on a cookie? Where is a Sector on a cookie?

Interpretations and Reminders

- Students are familiar with degree measure for angles. Teacher has to spend time on introducing radian measure.

Misconceptions

- When using the formula for arc length and area of sector, students may use the angle measure in degrees instead of radians. Constantly remind students to know the difference between the two.

Learning Progression (Suggested Learning Experiences)

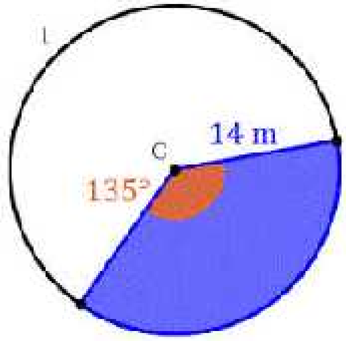
<u>Procedural Fluency:</u> (Recommended for 5 – 10 minutes each day: Fluency strategies are useful to activate student voice, solicit prior knowledge and develop fluency based on conceptual understandings.) For additional fluency practice strategies see the table at the end of this document.

Draw as many pictures as possible from your environment that displays sectors.

<u>Graduated Measure</u> (The graduated measure is a quick opportunity to diagnose students' level of comfort

Some of the language used in this document is adapted from the CCSS Progressions, NY, Utah, Arizona, Ohio, & Georgia state resources s.

with the material before you begin the progression.)
Student Directions: Select a question/problem from the table below that you feel best equipped to answer successfully.

Level 1	Level 2	Level 3
Find the area of the shaded region  	A lawn sprinkler located at the corner of a yard is set to rotate through 90° and project water out 30.0 feet. To three significant digits, what area of lawn does the sprinkler water?	Which Pizza is a better deal if bought by the slice?  

<u>Gradual Release of Responsibility</u>:

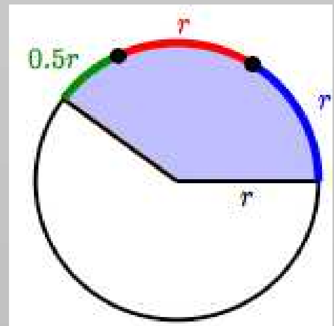
Focus Lesson 1

The definition of the ‘radian’ as a unit of measurement for angles and the formula for arc length, area of sector have all have to be done as an exploratory activity to promote conceptual understanding by students. Giving the definition and the formula and asking students to memorize and apply may not work well for retention in the long run. Providing students the opportunity to explore, discuss and learn will help in the retention and in understanding of concepts.

Ask each student to draw a circle and measure the radius using a string. Then place the string around the circumference marking the points as shown and ask students how many 'r's fit on the circumference? Is that an exact whole number? If there is a small portion left, what part of the radius is it?

If each 'r' corresponds to a central angle of 1 radian, how many radians are at the center of the circle?

Since you already know there are 360^0 at the center of the circle, what is the relationship between radians and degree measure?



The 'Sectors of Circles' Task from mathshell center can be utilized here. The task done with fidelity in the classroom will help students learn and master the standard in its entirety.

MAP-calculating arcs and areas of sectors

MAP-Arc lengths , Areas of sectors of ci

IM Orbitting satelite C 5.pdf

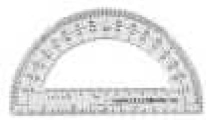
Collaborative Practice:

Shmoop Circles-sectors Works

Shmoop Circles-sectors Works

Additional Unit Assessments –				
Assessment Name	**Assessment**	**Assessment Type**	**Standards Addressed**	**Cognitive Rigor**
Two Wheels and a Belt	IM Circles Two wheels and a belt.pdf	Performance Task	MGSE9-12.G.C.5 MGSE9-12.G.SRT.8	DOK 3
Pizza Task	C 5- Pizza Sector Task.pdf	Performance Task	MGSE9-12.G.C.5	DOK 3

Differentiated Supports	
Learning Difficulty	• Provide students with kinesthetic activities and visual tools to understand concepts. • http://www.mathopenref.com/arcsectorarea.html -this web page has animations that helps in better understanding of concepts related to sectors. • While doing problems on paper, ask students to draw a picture and trace the arc length/ area using fingers or pencil to actually visualize what is being asked.
High-Achieving Students	IM Setting up sprinklers C5, SRT 8.p Even if students do not complete the calculations, the discussion on how to solve provides opportunities for students to gain mastery of the SMPs.
English Language Learners	• Have students write their understandings in a math journal. You may want to consider allowing EL students to write notes in their first language and annotate by identifying math specific words they do not have a translation for. • Allow students to create pictorial flash cards for new math words. • While doing problems on paper, ask students to draw a picture and trace the arc length/ area using fingers or pencil to actually visualize what is being asked.

Online/Print Resources	
Digital Resources	https://www.youtube.com/watch?v=enMRZOGKsCs https://www.youtube.com/watch?v=SjRQo7re5uw http://www.onlinemathlearning.com/area-sector.html https://learnzillion.com/lesson_plans/975-investigate-and-calculate-arc-length-by-using-circumference-and-ratios
Print Resources	 NY Engage C5 intro.pdf
Manipulatives/Tools	    Compass Protractor Ruler Sketchpad piece of string
Textbook Alignment	
Geometry, McGraw-Hill	pp. 706 - 714

Lesson Four Progression

Duration 6-7 days

Focus Standard(s)
Explain volume formulas and use them to solve problems **O** MGSE9-12.G.GMD.1 Give informal arguments for geometric formulas. a) Give informal arguments for the formulas of the circumference of a circle and area of a circle using dissection arguments and informal limit arguments. b) Give informal arguments for the formula of the volume of a cylinder, pyramid, and cone using Cavalieri's principle. **O** MGSE9-12.G.GMD.2 Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures. **O** MGSE9-12.G.GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. **Visualize relationships between two-dimensional and three-dimensional objects** **O** MGSE9-12.G.GMD.4 Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.
Performance-Based Objectives
As a result of their engagement with this unit…
MGSE9-12.G.GMD.2 – **SWBAT** make use of structure **IOT** create a formula for the circumference and area of a circle. **MGSE9-12.G.GMD.2** – **SWBAT** apply Cavalieri's principle **IOT** formulate an informal proof for the volume of a cylinder, pyramid, cone, sphere and other solids. **MGSE9-12.G.GMD.4** – **SWBAT** identify the shapes of two-dimensional cross-sections of three-dimensional objects **IOT** evaluate the volume of the solid. **MGSE9-12.G.GMD.4** – **SWBAT** identify three-dimensional objects generated by rotations of two-dimensional objects **IOT** derive formulas for volumes of solids. **MGSE9-12.G.GMD.3** – **SWBAT** apply and calculate volumes of cylinders, pyramids, cones, and spheres **IOT** solve problems.

Building Coherence

Across Grades:

6th Grade

Students found the surface area of right rectangular prisms.

7th Grade

Students found area and circumference of a circle, informal relationship b/w area and circumference, solved problems involving SA and Vol of prisms, drew cross section of 3D objects

8th Grade

Studentsused volume formulas for cones, cylinders and spheres to solve problems

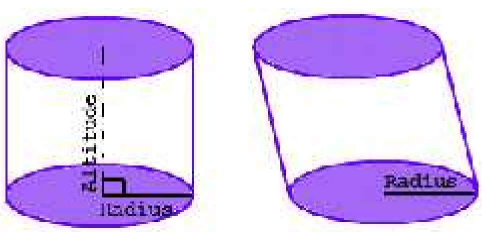
Within Grades:

Students not only apply volume formulas to solids, but also provide an informal proof for deriving the formula for volumes and apply the same to find volumes of composite solids.

Key Terms and Definitions

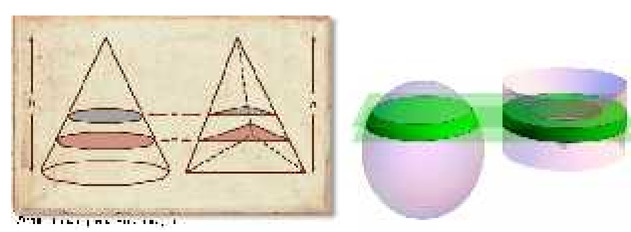
Cavalieri: Bonaven Francesco Cavalieri (1598-1647) was an Italian mathematician

Cavalieri's principle: If, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal.



The Cavalieri's principle can best be understood by looking at a stack of pennies. The volume of two stacks of the same number of pennies is the same, even if one stack is not vertically aligned.

Dissection: the action of dissecting a body or plant to study its internal parts. In this unit dissection arguments refer to the act of slicing various solids in order to find the shape and area of cross section at various heights of the solid.



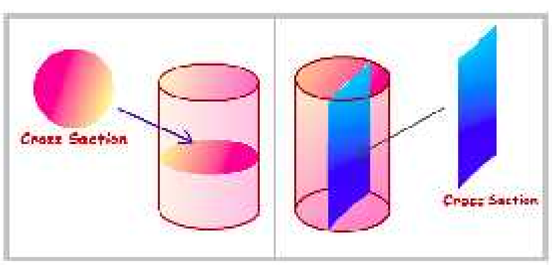
Limit: In **mathematics**, a **limit** is the value that a function or sequence "approaches" as the input or index approaches some value. **Limits** are essential to **calculus** (and **mathematical** analysis in general) and are used to **define** continuity, derivatives, and integrals.

(A simple explanation of limit: Consider a hot cup of boiling water taken from the microwave. Let T be the temperature of

Composite Figures: If a figure is made from two or more geometric figures, then it is called a Composite Figure.

Cone: A cone is a three-dimensional geometric shape that tapers smoothly from a flat base to a point called the apex or vertex

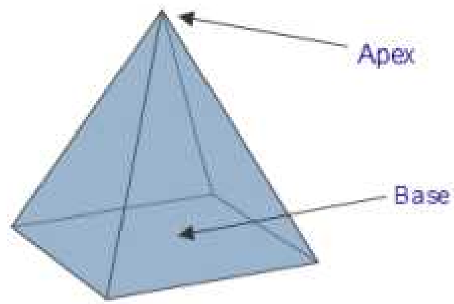
Cross section: A **cross section** is the **shape** we get when cutting straight through an **object**. The **cross section** of this **object** is a triangle. It is like a view into the inside of something made by cutting through it. This is a **cross**-**section** of a piece of celery.



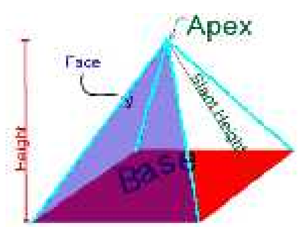
Cylinder: In its simplest form, a cylinder is the surface formed by the points at a fixed distance from a given straight line called the axis of the cylinder. It is one of the most basic curvilinear

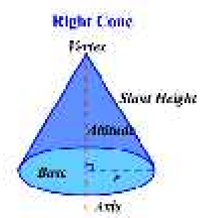
the water in the cup. Place the cup on the kitchen counter and examine T at various times. What is the limiting value of T? You observe that T decreases from 100^0 C as time moves on, but stops decreasing when it reaches the room temperature. So we can say the limiting value of T is the room temperature).

Pyramid: A **pyramid** is a polyhedron that has a base, which can be any polygon, and three or more triangular faces that meet at a point called the apex. These triangular sides are sometimes called the lateral faces to distinguish them from the base.

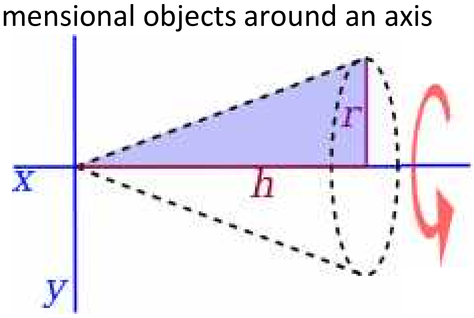


Slant Height: The diagonal distance from the apex of a right circular cone or a right regular pyramid to the base.

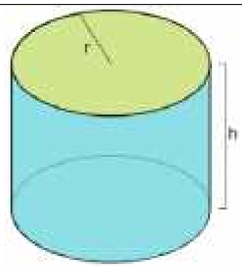




Solid of Revolution: The solid obtained by the rotation of a two-dimensional objects around an axis



Volume is the measure of the amount of space inside of a

 geometric shapes.	solid figure, like a cube, ball, cylinder or pyramid. It's units are always "cubic", that is, the number of little element cubes that fit inside the figure.

Guiding Questions

- How does Cavalieri's Principle apply to finding the Volume of a cylinder, even if it is oblique or not standing straight up?
- How do you think mathematicians found the formulas for finding volumes of prisms, cones and spheres?
- How can you identify the resulting solid of revolution when the graph of a simple function is rotated around an axis?

Interpretations and Reminders

- All the standards in this lesson progression demand the use of 2-D and 3-D models in order to gain conceptual understanding. If it is not possible to utilize models for teaching at all times, at least animated visuals need to be used to teach and learn concepts. A number of online free resources are provided in the unit.
- Providing opportunities for students to act and discover formulas like mathematicians facilitates in students appreciating the aesthetics in mathematics and enjoying the joy of doing mathematics!
- Students are already familiar with area and circumference of circle and volume of cylinder; however, they have internalized how it has been derived. It is necessary that students go through the process of proving informally using dissection method or infinite limits method.

Misconceptions

- An informal survey of students from elementary school through college showed the number pi to be the mathematical idea about which more students were curious than any other. There are at least three facets to this curiosity: the symbol π itself, the number 3.14159..., and the formula for the area of a circle. All of these facets can be addressed here, at least briefly.
- Many students want to think of infinity as a number. Avoid this by talking about a quantity that becomes larger and larger with no upper bound.
- The inclusion of the coefficient 1/3 in the formulas for the volume of a pyramid or cone and 4/3 in the formula for the volume of a sphere remains a mystery for many students. In high school, students should attain a conceptual understanding of where these coefficients come from.
- Concrete demonstrations, such as pouring water from one shape into another should be followed by more formal reasoning.
- While doing volume problems if the dimensions are given in different units, students often times ignore it and calculate volume. It has to be explained to students that units play an important role in finding volume.

Learning Progression (Suggested Learning Experiences)

Procedural Fluency: (Recommended for 5 - 10 minutes each day: Fluency strategies are useful to activate student voice, solicit prior knowledge and develop fluency based on conceptual understandings.) For additional fluency practice strategies see the table at the end of this document.

Draw each of the following solids. Then draw a horizontal cross section and vertical cross section in each case:

1. Cylinder 2. Cone 3. Sphere 4. Cube 5. Rectangular Prism 6. Tetrahedron 7. Triangular Prism

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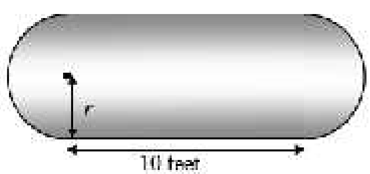
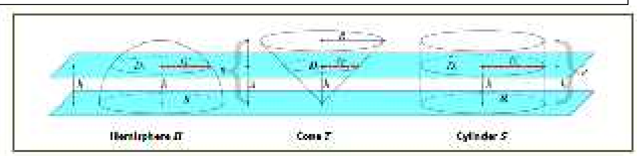
Graduated Measure (The graduated measure is a quick opportunity to diagnose students' level of comfort with the material before you begin the progression.)

Student Directions: Select a question/problem from the table below that you feel best equipped to answer successfully.

Level 1	Level 2	Level 3
Find the volume of a sphere with a diameter of 12 cm to one decimal place.	Find the radius of the tank that has twice the volume of the one given below:  Radius = 6 feet, height of the cylinder = 10 feet	 Use Cavalieri's principle to compare cross sectional areas of the three solids given above and hence calculate the volume of the hemisphere.

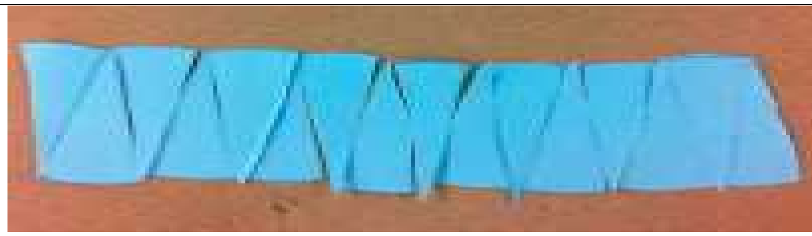
Gradual Release of Responsibility

Focus Lesson 1

The formula for area of a circle can be derived informally in two ways: 1. Using the dissection method of cutting the circle into thin sectors and rearranging to get a rectangle. 2. Finding the area of an octagon and using limits. A brief description of both methods are given below:

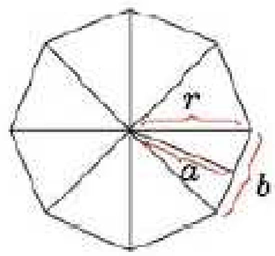
1. Provide each student with a paper circle. Ask students to fold it exactly on a diameter dividing the circle into two equal parts. Fold the circle 3 more times making sure the fold divides the existing shape into two equal halves. Open the circle to see 16 equal sectors. Now cut the individual sectors and rearrange the sectors to form a rectangle as shown below:

Some of the language used in this document is adapted from the CCSS Progressions, NY, Utah, Arizona, Ohio, & Georgia state resources s.



What is the approximate length of the rectangle? What is its width? What is the area if the shape is assumed to be a rectangle? (Do notice that increasing the number of sectors will make the shape a perfect rectangle).

2. Inscribe an octagon in a circle and find the area of the octagon.



The area of T is $\frac{1}{2}$base $\times$ height. In the picture, the base is b and the height is a. So the area of the regular octagon is

$$\begin{aligned} 8 \times \text{Area}(T) &= 8 \times \left(\frac{1}{2} b \times a\right) \\ &= (8 \times b) \times \frac{a}{2} \\ &= \text{Perimeter(Octagon)} \times \frac{a}{2}. \end{aligned}$$

b. We can decompose our regular polygon P_n with n sides into n congruent triangles as in part (a). We denote one of these triangles by T_n. We can then duplicate the reasoning at the end of part (a) to find the area of P_n. For this, we let b_n denote the length of the base of T_n and a_n its height.

$$\begin{aligned}\text{Area}(P_n) &= n \times \text{Area}(T_n)\\ &= n \times \left(\frac{1}{2} b_n \times a_n\right)\\ &= (n \times b_n) \times \frac{a_n}{2}\\ &= \text{Perimeter}(P_n) \times \frac{a_n}{2}.\end{aligned}$$

c. As the number of sides on our polygon P_n increases, the perimeter of P_n will approach the circumference of C while the distance a_n from the center of the circle to the sides of P_n will approach the radius r of C. Finally, the area of P_n approaches the area of C. Putting all of this together we get

$$\text{Area}(C) = \text{circumference}(C) \times \frac{r}{2}.$$

Since the circumference of C is $2\pi r$ this is the same as

$$\text{Area}(C) = 2\pi r \times \frac{r}{2} = \pi r^2.$$

Note: The above explanation is from the Illustrative Mathematics task 'Area of a circle'.

IM GMD 1 Area of a circle.pdf

Focus Lesson 2: Applying Cavalieri's principle to find volume of a cone, pyramid and a hemisphere.

IM GMD 1 Volumes of prism and cylinder usi

IM GMD4 Volume of special pyramid.pdf

IM GMD Use Cavalieri's Principle to

https://tapintoteenminds.com/3act-math/prisms-pyramids-3-act-math-task/

Guided Practice:

Identify the shapes of two-dimensional cross-sections of three-dimensional objects

Ask students to bring vegetables/fruits or any other object that are solid and could be sliced. Ask them to sketch the vertical/horizontal cross section in each case and name it if possible. Initiate a discussion on the cross section-if it the same size and shape throughout the entire length/ width. If changing, how does it change? For example, some apples are approximation of spheres and their horizontal cross section gives a circle that increases from the bottom, reaches a maximum and then decreases again. A horizontal cross section of a tomato is an approximate circle, whereas a vertical cross section is an approximation of an ellipse.

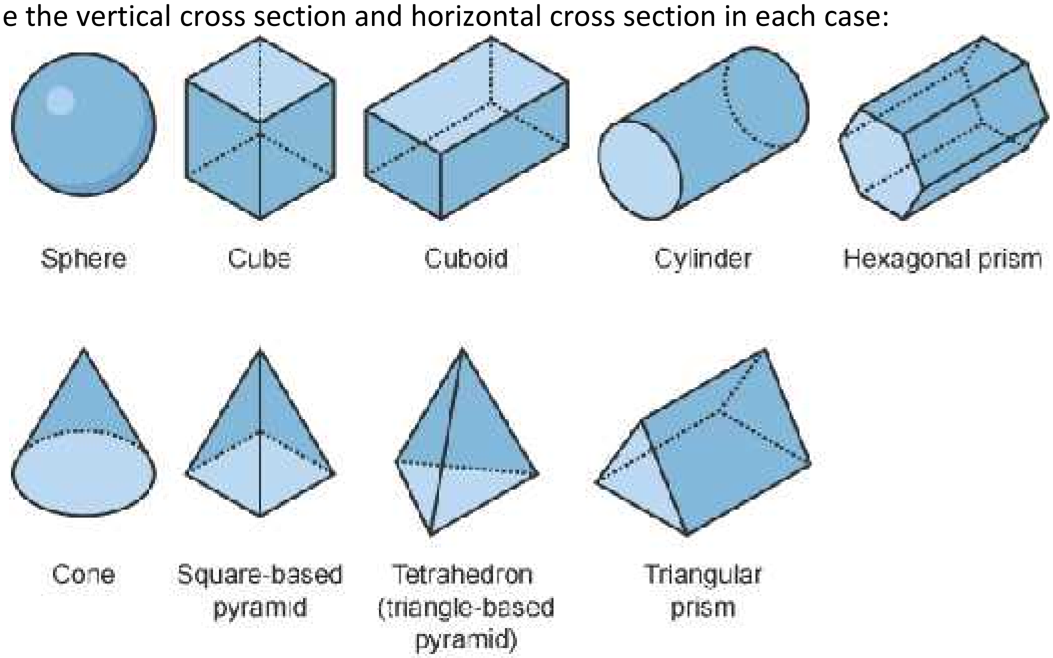
 

Once students are able to guess the cross section of known objects, the teacher can introduce other 3 d shapes not commonly found in the immediate home environment.

Rotation of 2D objects.pptx

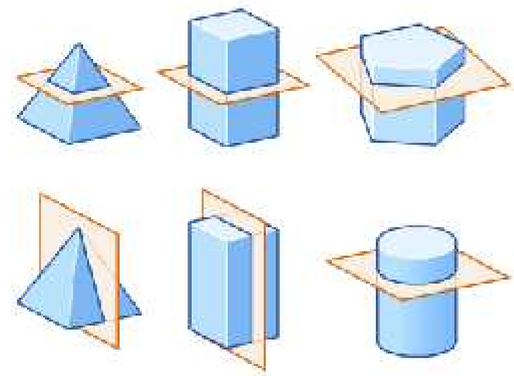
Collaborative practice:

Sketch and name the vertical cross section and horizontal cross section in each case:



Formative Assessment

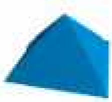
Name the cross section in each case:



Which of the objects shown on the right could be sliced to create square cross-sections?

Please note that the height of the cylinder is greater than the diameter of its base, the height of the cone is greater than the diameter of its base, and the length of the prism is greater than any side of its equilateral base.

Select all that apply.

- [ ]  Cone
- [ ]  Sphere
- [ ]  Cylinder
- [ ]  Cube
- [ ]  Triangular prism (equilateral base)
- [ ]  Right pyramid (square base)

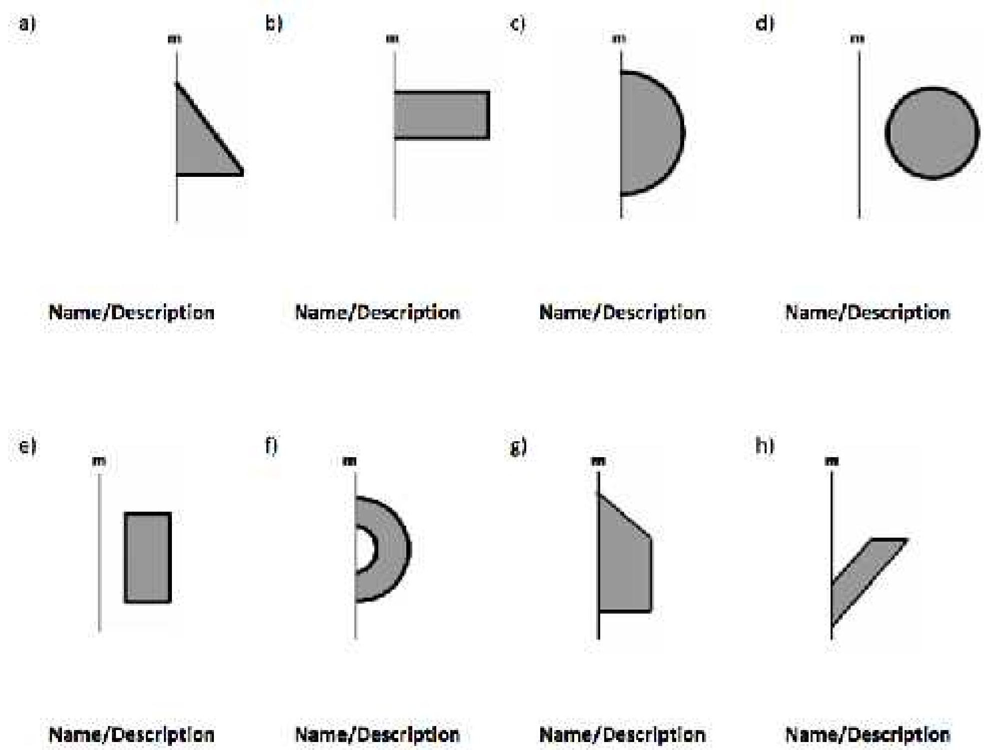
Focus Lesson 3:
Identify three-dimensional objects generated by rotations of two-dimensional objects

One of the best ways to visualize three-dimensional objects generated by two-dimensional objects is to use an animated video. The learnzillion lesson link given below does that for you:

https://learnzillion.com/lesson_plans/7269-predict-3d-results-of-rotating-simple-figures

NY Engage GMD 4 cross sections.pdf

1. Describe the solid that is formed by rotating each of these figures about line m and sketch it.



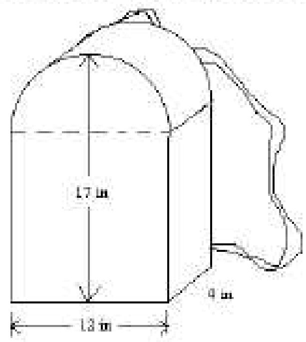
http://teacherweb.com/ID/Preston/Smith/gstnotes_g.gmd.4Wkst12.pdf

GMD 4 worksheet.pdf

Focus Lesson 4:
Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

Examples:

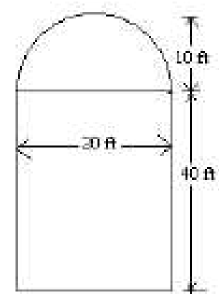
1. Approximate the Volume of the Backpack that is 17in x 12in x 4in.



SOLUTION

$$V = prism + \frac{1}{2}cylinder$$
$$= lwh + \frac{1}{2}(\pi r^2 h)$$
$$= ((17-6) \bullet 12 \bullet 4) + \frac{1}{2}(\pi 6^2)$$
$$= 528 + 18\pi\, in^3 \approx 584.55 in^3$$

2. Find the Volume of the Grain Silo shown below that has a diameter of 20ft and a height of 50ft.



SOLUTION

$$V = \frac{1}{2}sphere + cylinder$$
$$= \frac{1}{2}(\frac{4}{3}\pi r^3) + (\pi r^2 h)$$
$$= \left(\frac{2}{3} \bullet \pi \bullet 10^3\right) + \left(\pi \bullet 10^2 \bullet 40\right)$$
$$= \frac{2000}{3}\pi + 4000\pi\, ft^3$$
$$= \frac{14000}{3}\pi\, ft^3 \approx 14660.77\, ft^3$$

3. The diameter of a baseball is about 1.4 in.
   How much leather is needed to cover the baseball?
   How much rubber is needed to fill it?

SOLUTION

Leather to cover is SA

$$\begin{aligned} SA &= 4\pi r^2 \\ &= 4\pi \bullet .7^2 \\ &= 1.96\pi\, in^2 \approx 6.16 in^2 \end{aligned}$$

Rubber to fill is V

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi \bullet .7^3 \\ &= \frac{343}{750}\pi\, in^3 \approx 1.44 in^3 \end{aligned}$$

4. The volume of a cylindrical watering can is 100cm^3. If the radius is doubled, then how much water can the new can hold?

SOLUTION

$V = \pi r^2 h$

$V_{new} = \pi (2r)^2 h$

$V_{new} = 4\pi r^2 h$

$V_{new} = 4(old\ volume)$

$V_{new} = 4 \bullet 100 = 400 cm^3$

It is important to note that since only the radius is doubled and not the height, you cannot just multiply by 2^2.

Guided Practice: Volumes of solids

IM GMD 3
Centerpiece.pdf

Collaborative Practice: Three act task-Meatballs

In this three-act task, students use volumes to calculate how many meatballs can be dropped into the sauce without making the sauce overflow. Students are engaged and discuss about the thickness of the sauce, rim of the vessel, etc.

http://www.101qs.com/2352-meatballs

Additional Unit Assessments –				
Assessment Name	**Assessment**	**Assessment Type**	**Standards Addressed**	**Cognitive Rigor**
Tennis Ball in a can	IM GMD Tennis ball in a can.pdf	Performance Task	GMD 4	DOK 3
Doctor’s Appointment	IM GMD Doctor’s appointment.pdf	Performance Task	GMD 3	DOK 3
Best Size Cans	MAP bestsize_cans.pdf	Performance Task	GMD 3	DOK 3

Differentiated Supports	
Learning Difficulty	• Students may have difficulty understanding the dimension of an object. Ask them to visit this webpage before the lesson. • http://www.amnh.org/ology/features/stufftodo_einstein/threed_dimension.php
High-Achieving Students	Challenge gifted and high achieving students with the following formative assessment lesson: MAP propane_tanks.pdf
English Language Learners	• Have students touch and feel different shapes to make connection to the new shapes they are not familiar with. • You may want to consider allowing EL students to write notes in their first language and annotate by identifying math specific words they do not have a translation for. • Allow students to create pictorial flash cards for new math words. • Provide students with visual proofs for volumes of various solids.

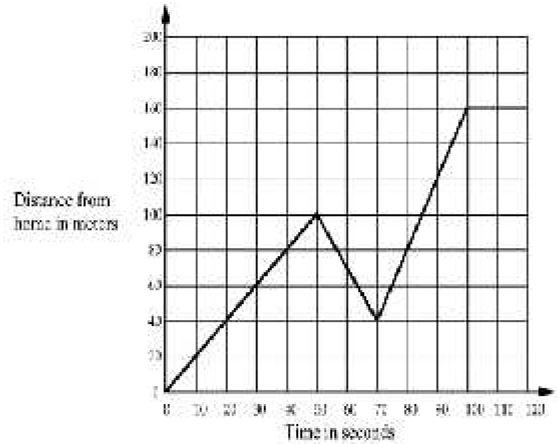
Online/Print Resources	
Digital Resources	• https://schoolyourself.org/learn/geometry/cavalieri-3d • http://math.tutorvista.com/geometry/cavalieris-principle.html • https://betterexplained.com/articles/an-intuitive-introduction-to-limits/ • http://www.cpalms.org/Public/PreviewResourceAssessment/Preview/56776

	• http://www.onlinemathlearning.com/cross-section-3d-shapes-hsg-gmd4.html • http://www.schoolmath.jp/3d/e/student/lesson01/lesson_02.htm • http://www.sparknotes.com/testprep/books/sat2/math1c/chapter7section5.rhtml • http://www.ck12.org/geometry/Solids-Created-by-Rotations/lesson/Connections-Between-Two-and-Three-Dimensions-GEOM-HNRS/
Print Resources	MAP-calculating volumes of compounc MAP bestsize_cans.pdf
Manipulatives/Tools	 Compass  Protractor  Ruler  Sketchpad
Textbook Alignment	
Geometry, McGraw-Hill	pp. 798, 839, 863 – 880

Sample Fluency Strategies for High School

Expression of the Week	Choose an expression for the week. Allow students to consider the given expression. Students will then think of different ways to rewrite that expression in order to create equivalent expressions. As course concepts progress ask students to incorporate additional concepts in the creation of their expression. Provide students an opportunity to explain their thinking.
Algebraic Talks	Extending from the idea of Number Talks. Teachers can help students develop flexibility through Algebraic Talks. **<u>Algebraic Talks (The Norms):</u>** - No Pencils - No calculators - Mental Math Only - Consider the problem individually during the first few minutes Place a problem on the board and allow students to consider possible solution pathways. Next, allow 5 or 6 student to explain their strategy and record the various methods on the board for all students to see. Name the strategy (preferably after the student who made the conjecture). Allow other students to ask questions about the strategies while the proposing student is allowed to defend his/her work. Also see: Jo Boaler Number Talks Videos
Picture This	Provide students with a graph, or chart and ask them to come up with a contextual situation that makes the picture relevant

Every morning Tom walks along a straight road from his home to a bus stop, a distance of 160 meters. The graph shows his journey on one particular day



Find the odd man out

A	**B**
x – y = 5	**2x + 3y = 6**
C	**D**
2x – 3y = 0	**3y = 2x – 4**

Note 1: This is an open ended question where students can choose any one of the choices A-D as the odd man as long as they are able to defend their choice. For example,
A is the odd equation because it is in the only equation with coefficients as 1 or -1
B is the odd equation because it is in the only equation with negative slope
C is the odd equation because it is in the only equation whose line passes through the origin
D is the odd equation because it is in the only equation which is not represented in the standard form.

Note 2: Teacher can provide 4 equations/ 4 triangles / 4 transformations / 4 trig ratios where there is more than one way of classifying. Students can choose the odd man by mere inspection (surface level) or by examining the four components with a deeper understanding of the concept.

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Example 2:
<table>
<tr><td>A
$y = x^2+5x+6$</td><td>B
$y = x^2- 5x+6$</td></tr>
<tr><td>C
$y = x^2- x-6$</td><td>D
$y = - x^2 + 6$</td></tr>
</table>

A is the odd equation because it has all positive coefficients or because it has two negative roots.

B is the odd equation because it has two positive roots.

C is the odd equation because it has one positive and one negative root.

D is the odd equation because the parabola opens downwards
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<td>Generate examples and non-examples</td>
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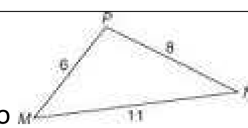
Example 1: Draw as many triangles as possible that are similar to 

Note: Teachers can ask students to generate examples and non-examples for a variety of topics to improve fluency. A few more possibilities are listed here:
1: Create quadratic equations with imaginary roots
2. Create quadratic equations with complex roots
3. Create non-examples of complementary angles.
4. Create examples of circles whose center is (3, -2)
5. Draw triangle ABC where Sin A= 0.3
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<td>Locating the error</td>
<td>Choose a student’s work to be displayed on board after removing the name and ask the class to locate the error and also work the solution correctly. This can be done in any topic where the problem requires multiple steps to solve.</td>
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